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Matrix Cosmology¹

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Abstract

Some speculative preliminary ideas relating matrix theory and cosmology are discussed.

¹Slightly extended version of the talk given by G. W. Gibbons at the Mitchell Institute Conference on Cosmology

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1 Motivation

This is a report on some on-going work in which an attempt is made to explore how to incorporate the basic ideas of cosmology into M-Theory. It may be seen either in the context of much recent work on time dependent backgrounds in String Theory, or in its own right, as a speculative approach to cosmology aimed at ultimately taking us beyond the standard Friedman-Lemaitre paradigm. More concretely, our motivations are

- The BFSS matrix model [1] is claimed to provide a fundamental quantum mechanical description of “M-Theory”².
- It replaces conventional spacetime concepts, such as *commuting* coordinates, with inherently non-classical notions such as *non-commuting* coordinates.
- It should, therefore, surely have something deep to say about the structure of the universe.

²For a useful review of M(atrrix) theory see e.g. [2].

- In particular one should be able to use it to address such issues as the existence and significance of such things as “The Wave function of the Universe”.

In this talk we present some *rudimentary* and very *preliminary* ideas aimed at understanding how we should think about cosmology in the language of matrix theory. It is a report of work done partly in collaboration with S. Alexander. The only previous work on this topic known to us is that of Alvarez and Meessen [3].

2 Matrices and D0-particles

One may view the matrix model in two slightly different ways.

- Following BFSS, as the limit $N \rightarrow \infty$ of a super-quantum-mechanics of 9 $N \times N$ Hermitian matrices.
- Following earlier work by de Wit, Hoppe and Nicolai [4], as a regularization of the super-membrane of 11-dimensional supergravity.
- Both approaches lead, because of the high amount of super-symmetry, to 10-dimensional super-Yang-Mills with gauge group $G = U(N)$ and fermions in the adjoint representation reduced to one spacetime dimension. In Coulomb gauge, one replaces the $\mathfrak{u}(N)$ valued connection one-forms $A_\mu(\mathbf{x}, t)$ by their 9 spatial components $A_i(t)$ which are the 9 Hermitian matrices $X^i(t)$ of the model.
- From the membrane point of view one passes to light-cone gauge and the X^i represent the 9 transverse components of the membrane coordinates. The residual bosonic gauge-invariance consists of $\text{sdiff}(\Sigma_2)$, area preserving diffeomorphism of the membrane 2-manifold Σ_2 . The Lie algebra of $\text{sdiff}(\Sigma_2)$ is well known to coincide, in some sense at least, with $\lim_{N \rightarrow \infty} \mathfrak{u}(N)$.
- From the 10-dimensional String Theory point of view one should regard the X^i as representing the 9 non-commuting position coordinates of N D0-branes, the locations of the ends of fundamental strings.
- *Clusters* of large numbers of D0-particles are described by classical solutions of 10-dimensional Type IIA supergravity theory.

The BPS states correspond to electrically charged singular extreme “black hole” hole solutions where the electric Ramond-Ramond charge couples to a graviphoton field of the 10-dimensional Type IIA super-gravity theory which may be obtained by dimensional reduction of the 11-dimensional super-gravity theory.

The classical solutions, which describe k separated clusters in force balance may be lifted to 11-dimensions where they have the structure of k singular vacuum pp-wave solutions moving parallel to each other. The pp-waves are sometimes described as 11-dimensional gravitons, but this is not really accurate, because even in 11-dimensions, the solutions have distributional sources. A better description is as lightlike cylinders extending along the 10'th spatial dimension.

3 Classical Matrix Theory

One approach to Matrix Cosmology is via the classical equations of motion [3]. We shall briefly describe this, since it was the original approach that we adopted, but later we will, for reasons to be explained, abandon it for a rather different picture.

The basic classical equations of matrix theory are

$$\boxed{\frac{d^2 X^i}{dt^2} + [X^j, [X^j, X^i]] = \lambda X^i,} \quad (1)$$

where X^i are $n \times n$ hermitian matrices, the index j is summed over and $\lambda = \frac{\Lambda c^2}{3}$ is a possible cosmological or mass term [5, 3, 6, 7].

In the BFSS model $n = 9$, and one is looking at N D0-branes but these equations have been studied more widely as a reduction of $U(N)$, or, if they are taken traceless, $SU(N)$, Yang-Mills theory to one time dimension. As such, there is some evidence for chaotic behaviour.

Note that

- If $\lambda = 0$, we have a Galilei invariant system
- If $\lambda \neq 0$, we have invariance under one of the two Newton-Hooke groups, of the two non-relativistic contractions of the De-Sitter ($\lambda > 0$) or Anti-de-Sitter ($\lambda < 0$) groups. A description of these groups and their transformation rules together with an account of their significance for Newtonian cosmology with a cosmological constant are given in [7].
- If one thinks in terms of a mass term, then a positive mass squared corresponds to negative cosmological constant and a tachyonic mass term to a positive cosmological constant.
- In the case of the BFSS model, the equations of motion must be supplemented by a *constraint* on the initial conditions which arises from the Gauss constraint of the gauge theory

$$[\dot{X}^i, X^j] = 0. \quad (2)$$

4 Newtonian Cosmology

Since our equations lack manifest covariance, the most helpful analogy is with elementary Newtonian Cosmology in 3 spatial dimensions. The brief presentation of Newtonian cosmology which follows may be unfamiliar, but it is completely equivalent to more conventional accounts in the literature. The generalization to other space dimensions, and indeed to other force laws, is trivial.

Newton's equations of motion for k gravitating particles are

$$\boxed{m_a \ddot{\mathbf{r}}_a = \sum_b G \frac{m_a m_b (\mathbf{r}_b - \mathbf{r}_a)}{|\mathbf{r}_a - \mathbf{r}_b|^3} + \lambda m_a \mathbf{r}_a.} \quad (3)$$

- As with the matrix model, we either have Galileo invariance ($\lambda = 0$) if the cosmological constant vanishes, or if it does not we have Newton-Hooke invariance.
- In order to incorporate the *Cosmological Principle* we make a *Homothetic Ansatz*

$$\mathbf{r}_a(t) = a(t)\mathbf{x}_a, \quad (4)$$

where the so-called *co-moving coordinates* \mathbf{x}_a are independent of time.

The homothetic ansatz leads to two conditions.

- *Raychaudhuri's Equation*

$$\frac{\ddot{a}}{a} = -\frac{\mu}{3a^3} + \lambda. \quad (5)$$

This is the usual equation of motion for the scale factor of an expanding universe with a cosmological term and pressure free fluid. In what follows we set $\lambda = 0$ for simplicity. It is easy to adapt the discussion to the case $\lambda \neq 0$ (see [7]).

- The co-moving coordinates must constitute a *Central Configuration*, i.e. a solution of

$$\frac{\mu}{3}m_a\mathbf{x}_a + \sum G \frac{m_a m_b (\mathbf{x}_b - \mathbf{x}_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3} = 0. \quad (6)$$

The quantity μ is a constant. Central configurations are extrema of an auxiliary potential

$$\sum \frac{\mu}{6}m_a\mathbf{x}_a^2 + \sum \sum G \frac{m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \quad (7)$$

Recently, with Battye and Sutcliffe [8], one of us has carried out an extensive numerical investigation of central configurations which are minima or *ground-states* of this potential for up to 10^4 particles. In the case of equal masses $m_a = m$, $\forall a$, the conclusion is that the minima correspond to a spherical ball of particles of uniform density $\frac{\mu}{4\pi G}$. In other words, if N is the total number of particles and the radius ρ is defined by

$$\frac{GNm}{\rho^2} = \frac{\mu\rho}{3}, \quad (8)$$

then one finds a uniform density of particles inside the radius ρ and almost no particles outside that radius. The interpretation of (8) should be clear. It is well known that a spherical shell of matter exerts no force on particles inside it but an attraction on particles outside given by the total mass of the shell. The left hand side of (8) is the Newtonian attraction due to the total mass Nm interior to radius ρ on the thin shell of particles at ρ . The right hand side is the repulsive pseudo-cosmological force on the thin shell which is proportional to the distance ρ . We use the term ‘pseudo-cosmological’ to alert the reader to the fact that we get such a term even if the cosmological constant $\lambda = 0$. It really arises from the inertial term in the Newtonian equation of motion (3). Note that the proper radius of our ball is time-dependent and given

$$|\mathbf{r}| = R = a(t)\rho. \quad (9)$$

A ball of uniform density is exactly what one expects on the basis of the usual pressure free fluid model. Thus this, slightly unconventional, approach to Newtonian Cosmology reproduces all of the standard features without making arbitrary assumptions about fluids, rather these assumptions are *derived* from the model.

4.1 Quantum Newtonian Cosmology

In order to prepare ourselves for Quantum Matrix cosmology, it may be worth pausing to recall that one can obviously construct a Wave Function for Newtonian Cosmology in the framework of non-relativistic quantum mechanics. This may not often be done in discussions of Quantum Cosmology but it is entirely straightforward and elementary. All that one needs is a solution of the multi-particle Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \sum -\frac{\hbar^2}{2m_a} \nabla_a^2 \Psi + V\Psi, \quad (10)$$

with $\Psi = \Psi(\mathbf{r}_a)$, $\nabla_a^2 = \frac{\partial^2}{\partial \mathbf{r}_a^2}$ and

$$V = - \sum \sum G \frac{m_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|}. \quad (11)$$

4.2 WKB Approximation

We consider a potential $V = V(\mathbf{r}_a)$ which is homogeneous of degree n . For the case of Newtonian gravity without a cosmological term corresponds to $n = -1$. The equation of motion is

$$m_a \ddot{\mathbf{r}}_a = - \frac{\partial V}{\partial \mathbf{r}_a}. \quad (12)$$

The homothetic ansatz is

$$\mathbf{r}_a = a(t) \mathbf{x}_a, \quad (13)$$

where the co-moving coordinates \mathbf{x}_a constitute a *central configuration* satisfying

$$\frac{\mu}{3} m_a \mathbf{x}_a = + \frac{\partial V}{\partial \mathbf{x}_a} \quad (14)$$

and the scale factor satisfies the Raychaudhuri type equation

$$a^{1-n} \ddot{a} = - \frac{\mu}{3}. \quad (15)$$

with first integral or Friedmann equation

$$\frac{1}{2} \dot{a}^2 + \frac{1}{n} \frac{\mu}{3} a^n = \frac{k}{2}, \quad (16)$$

where k is a constant. Taking the dot product of \mathbf{x}_a with (14) and using Euler's theorem gives the Virial Theorem

$$\frac{\mu}{3} \sum m_a \mathbf{x}_a^2 = nV(\mathbf{x}_a). \quad (17)$$

The conserved energy is

$$H = \frac{k}{2} \sum m_a \mathbf{x}_a^2. \quad (18)$$

At the JWKB level, the wave function is

$$\Psi \approx e^{\frac{iS}{\hbar}}, \quad (19)$$

where S is the relevant solution of the Hamilton-Jacobi equation

$$\sum \frac{1}{2m_a} \left(\frac{\partial S}{\partial \mathbf{r}_a} \right)^2 + V(\mathbf{r}_a) = -\frac{\partial S}{\partial t}. \quad (20)$$

In our case the relevant solution is

$$S = \sum \frac{\dot{a}}{a} \frac{m_a}{2} \mathbf{r}_a^2 = \sum a \dot{a} \frac{m_a}{2} \mathbf{x}_a^2 = a \dot{a} \frac{H}{k}. \quad (21)$$

4.3 Hartree-Fock approximation

Here we suppose all masses equal $m_a = m$ and replace the full wave function $\Psi(\mathbf{r}_a)$ by the product

$$\Psi(\mathbf{r}_a) \propto \prod_a \Psi'(\mathbf{r}_a), \quad (22)$$

where $\Psi'(\mathbf{r})$ satisfies

$$i\hbar \frac{\partial \Psi'}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi' + mU \Psi', \quad (23)$$

with $\Psi' = \Psi'(\mathbf{r}, t)$, $\nabla^2 = \frac{\partial^2}{\partial \mathbf{r}^2}$ and

$$\nabla^2 U = 4\pi G m |\Psi'|^2. \quad (24)$$

The time-independent Schrödinger equation coupled to Poisson's equation has been studied in a different context where it is referred to as the Schrödinger-Newton equation [9].

In our case, we assume that

$$U = F(t) \mathbf{r}^2, \quad \Psi' = A(t) e^{\frac{iS}{\hbar}}, \quad (25)$$

We find from the Poisson's equation that

$$A^2 = \frac{3F}{2\pi G m}. \quad (26)$$

One readily sees that one must have

$$S = B(t) \mathbf{r}^2, \quad (27)$$

with

$$-\frac{3B}{m} = \frac{\dot{A}}{A}. \quad (28)$$

Moreover if $A = a^{-\frac{3}{2}}$, then the scale factor $a(t)$ satisfies the Raychaudhuri equation

$$\ddot{a} = -\frac{4\pi Gm}{3a^2}. \quad (29)$$

The action is given

$$S = \frac{1}{2}m\frac{\dot{a}}{a}\mathbf{r}^2, \quad (30)$$

and the wave function by

$$\boxed{\Psi' \propto \frac{1}{a^{\frac{3}{2}}} e^{\frac{i}{2\hbar}m\frac{\dot{a}}{a}\mathbf{r}^2}.} \quad (31)$$

In this case the Hartree-Fock approximation gives a version of the WKB wave function corrected by the prefactor $\frac{1}{a^{\frac{3}{2}}}$.

4.4 Normalization and Energy

In order to model the central configurations described earlier which have a finite number of particles, we need to use a normalizable wave function. We use the Hartree-Fock wave function $\Psi'(\mathbf{r})$ of (31) in the region $|\mathbf{r}| < R$ and take $\Psi'(\mathbf{r}) = 0$ for $|\mathbf{r}| > R$. This is an exact solution of the Schrodinger equation (23) within each region, but fails at the surface $|\mathbf{r}| = R$. We ignore this issue here.

The normalization integral of $|\Psi'(\mathbf{r})|^2$ has support in a ball of proper radius $|\mathbf{r}| = R$. The norm must be time-independent, so we need to take *time dependent* $R(t)$. In fact we need to take

$$R(t) \propto a(t) \quad (32)$$

This is consistent with the classical analysis giving Hubble's law (9).

The classical problem of Newtonian cosmology has a conserved energy, and we should check the energetics of our quantum-mechanical model. The energy of a single particle wave function is

$$E = \int d^3r \left[\frac{\hbar^2}{2m} |\nabla \Psi'|^2 + mF(t)\mathbf{r}^2 |\Psi'|^2 \right] \quad (33)$$

$$\propto \left[\frac{2}{m} B(t)^2 + F(t) \right] A(t)^2 \int^{R(t)} d^3r \mathbf{r}^2 \quad (34)$$

$$\propto \frac{1}{2} \dot{a}(t)^2 + \frac{2\pi Gm}{3a(t)}. \quad (35)$$

The Raychaudhuri equation (29) has the first integral

$$\frac{1}{2} \dot{a}(t)^2 - \frac{4\pi Gm}{3a(t)} = \frac{1}{2} k, \quad (36)$$

so we have

$$E \propto \frac{1}{2}k + \frac{2\pi Gm}{a(t)}. \quad (37)$$

This is a constant, as desired, plus a t -dependent error term which we attribute to the sharp cutoff in the wave function.

Since the quantum-mechanical energy (33) is strictly positive, we must choose the $k > 0$ solution of (29) with large t behavior $a(t) \sim kt$. The error term above vanishes at large t . The quantum mechanical model is thus consistent for an “open universe”.

4.5 Wick Rotated Newtonian Wave function of the Universe

We have constructed an approximate wave function of a simple Newtonian universe whose WKB approximation gives a classical solution of Newton’s equations of motion representing an expanding gas of point particles. The aim of quantum cosmology is to derive this wave function, and hence the initial conditions for the universe from some more fundamental assumption, such as the No-Boundary Proposal of Hartle and Hawking. We shall not dwell on this in detail here but content ourselves with the following, possibly suggestive, remark. If we take the simplest (Einstein-de-Sitter) solution for the scale factor $a(t) \propto t^{\frac{2}{3}}$, we have

$$\Psi' \propto e^{\frac{i}{3\hbar} \frac{m\pi^2}{t}}. \quad (38)$$

Curiously, this Euclidean wave function, strictly speaking a solution of the diffusion equation rather than the Schroedinger, will be normalizable with respect to integrations over the positions if we Wick rotate, i.e. set

$$t = -i\tau, \quad (39)$$

with the imaginary time coordinate τ being real and positive. One might speculate that this normalizability of the Wick-rotated Newtonian wave function is related to Hartle and Hawking’s path integral approach to the wave function of the universe.

5 Homothetic Matrix Cosmology

After preparing ourselves with a brief excursion into Newtonian Cosmology, we return to the matter at hand. In matrix cosmology is it also natural to begin by making a homothetic ansatz [3, 10]

$$X^i = a(t)M^i, \quad (40)$$

where M^i are independent of t .

Substitution leads to

$$\frac{\ddot{a}}{a^3} - \frac{\lambda}{a^2} = \mu, \quad (41)$$

$$\mu M^i + [M^j, [M^j, M^i]] = 0. \quad (42)$$

The idea is to interpret (41) as the analogue of Raychaudhuri's equation in cosmology and (42) as the analogue of the equation governing central configurations in Newtonian cosmology [8] or monopole scattering [11]. Essentially the same equation arises in supersymmetric $\mathcal{N} = 1^*$ gauge theories when one is looking for vacua or ground states [12]. For that reason we shall sometimes refer to solutions of (42) as vacua. Similar equation also arises in certain solutions describing spinning membranes [13].

We begin by looking at Raychaudhuri's equation (41). It has a first integral

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\mu a^2}{2} + \lambda, \quad (43)$$

where k is a constant of integration. This is the analogue of the Friedman equation in standard cosmology. As well as a standard cosmological constant or dark energy contribution given by λ we have an exotic matrix contribution to the energy density given by

$$\rho_M = \frac{3\mu a^2}{16\pi G}. \quad (44)$$

If μ is positive this energy density *increases* as the universe expands, indicating that the pressure P_M has the opposite sign to the energy density. In d spatial dimensions this would lead to

$$P_M = -\frac{d+2}{d}\rho_M. \quad (45)$$

From the point of view of matrix theory the most natural choice for d would be 9. Arguing by analogy with the Newtonian case one might then regard $a(t)$ as the scale factor in Einstein conformal gauge. This then leads to

$$P_M = -\frac{11}{9}\rho_M. \quad (46)$$

Later we shall compare this with a model based on a supergravity solution representing a gas of expanding D0-branes.

As an example of the general theory, consider 3 $N \times N$ matrices M^i , $i = 1, 2, 3$ providing an N dimensional representation of $\mathfrak{su}(2)$,

$$[M^1, M^2] = iM^3 \quad \text{etc.} \quad (47)$$

This solution actually describes an expanding spherical membrane (see [2] and references therein). From (42) we find

$$\mu = -2, \quad (48)$$

which implies a *negative* energy density and positive pressure. This looks rather unphysical and so we turn to an anisotropic model. Recall that

$$9 = 3 + 3 + 3, \quad (49)$$

and take 3 mutually commuting sets of such matrices, each with its own scale factor $a(t), b(t), c(t)$ say. If M^1, M^2, M^3 are taken to be diagonal and $\lambda > 0$ we shall get exponential expansion for the scale factor a since

$$\ddot{a} = \lambda a. \quad (50)$$

On the other hand, we can have $X^4, X^5, \dots X^9$ oscillating. In other words

$$\boxed{\text{3 directions exponentially expand}} \quad (51)$$

$$\boxed{\text{6 directions oscillate}} \quad (52)$$

This phenomenon is closely related to the well-known *chaotic behaviour* of Yang-Mill reduced to one time dimension and zero space dimensions.

5.1 Chaos

In standard $\mathfrak{su}(2)$ Yang-Mills, one may assume that the connection is

$$A = X^i dx^i = u(t)\tau^1 dx + v(t)\tau^2 dy + w(t)\tau^3 dz, \quad (53)$$

where τ^i are Pauli matrices. The equations of motion derive from the Lagrangian

$$L = \frac{1}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) - V(u, v, w), \quad (54)$$

with

$$V(u, v, w) = \frac{1}{2}(u^2 v^2 + v^2 w^2 + w^2 u^2). \quad (55)$$

The non-negative potential V has *three commutative valleys* along the three orthogonal coordinate axes in u, v, w space for which V vanishes. Studies of the motion [14] show that the representative particle rattles along each valley, eventually returning to the origin and rattling along another valley. This is rather reminiscent of the behaviour of the three scale factors a, b, c of a Bianchi IX chaotic cosmology of the type originally studied by Misner [15]. Introduction of a positive cosmological term, (i.e. a tachyonic Higgs mass) leads to the *eventual escape of the particle in one direction*, provided Λ exceeds a certain threshold.

An obvious extension of this idea is to consider an $SU(2) \times SU(2) \times SU(2)$ model in which *just three directions expand exponentially* and the other six remain bounded.

6 D0-particle Cosmology

According to Type IIA ten-dimensional Supergravity, D0-branes correspond to extreme black holes, a static configuration of k clusters depending on a *harmonic function* H on \mathbb{E}^9

$$H = 1 + \sum_{a=1}^k \frac{\mu_a}{7|\mathbf{x} - \mathbf{x}_a|}, \quad (56)$$

where μ_a is quantized, being proportional to N_a for N_a D0-branes located at positions \mathbf{x}_a .

The moduli space is clearly given by k points in \mathbb{E}^9 . The slow motion is governed by a metric induced from the De-Witt metric of the Type IIA action. Long ago, Shiraishi showed that this metric is flat [16]. In other words, there are no velocity dependent forces quadratic in velocity. Thus one might anticipate that a cosmology of D0-branes should expand freely like a *ten-dimensional* Milne model which has

$$a(t) \propto t, \quad \rho + 9P = 0. \quad (57)$$

This is contradicted by some exact solutions of Type IIA found by Maki and Shiraishi [17] some time ago, following earlier work by Kastor and Traschen. In these one has

$$a(t) \propto t^{\frac{1}{5}}, \quad \rho = P, \quad (58)$$

which corresponds to ‘stiff matter’, for which the sound speed is that of light. Neither of these two equations of state coincides with that given by the homothetic matrix model of the previous section. One might try, as was suggested to us by Justin Khoury to ‘save appearances’ by passing to string conformal gauge, which might seem more appropriate for the homothetic matrix model scale factor. However, in this gauge, we would have

$$P = -\frac{1}{3}\rho, \quad (59)$$

which does not coincide with the $P = -\frac{11}{9}\rho$ which we obtained from the homothetic matrix model. In hindsight this is perhaps not so surprising, since the matrix configuration leading to this equation of state were rather delocalized and in fact corresponded to extended objects. For well localized D-particles with mutually commuting coordinates the quartic term in the matrix Lagrangian is effectively zero and therefore the dominant interaction between D-particles will come from a 1-loop correction which generates a term v^4/r^7 in the 2-body Lagrangian. It would be interesting to see whether Newtonian approach to cosmology based on those interactions could generate Maki-Shiraishi solutions. We shall not explore that in detail here but rather we shall describe the Maki-Shiraishi metrics and their ‘hidden supersymmetry’. There is no denying that a cosmology made up entirely of D0-branes, with no anti-D0-branes might not be thought of as being very realistic. Nevertheless, the solutions we are about to discuss do exhibit some extremely interesting features which we hope contains lessons for future, more realistic models.

6.1 Maki-Shiraishi metrics

Maki and Shiraishi [17] considered as a Lagrangian in $n+1$ spacetime dimensions for gravity plus a two-form plus a scalar

$$L = R - \frac{4}{n-1}(\nabla\phi)^2 - e^{\frac{4a}{n-1}\phi}F^2 - (n-1)e^{\frac{4b}{n-1}\phi}\Lambda. \quad (60)$$

where a and b (and Λ) are constants.

They sought solutions of the form

$$ds^2 = -H^{-\frac{2(n-2)}{n-2+a^2}} dt^2 + a^2(t) H^{\frac{2}{n-2+a^2}} d\mathbf{x}^2. \quad (61)$$

$$H = 1 + \frac{1}{a(t)^{n-2+\frac{p}{2}}} \sum \frac{\mu_a}{(n-2)|\mathbf{x} - \mathbf{x}_a|^{n-2}}, \quad (62)$$

$$e^{\frac{4a}{n-1}\phi} = a(t)^p H^{-\frac{2a^2}{n-2+a^2}}, \quad (63)$$

$$A = \sqrt{\frac{n-1}{2(n-2+a^2)}} \frac{dt}{a(t)^{\frac{p}{2}}} \left(1 - \frac{1}{H}\right), \quad (64)$$

with $F = dA$. Maki and Shiraishi found various solutions. The time dependence of the scale factor depends on the particular solution, for us the relevant one satisfies $n = a^2 = 9$, $p = 2a^2$, $\Lambda = 0$ and

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{9}}. \quad (65)$$

If no D0-branes are present the *background metric* is

$$ds_{10}^2 = -dt^2 + a^2(t) d\mathbf{x}^2, \quad (66)$$

with

$$g = e^\phi = t^{\frac{4}{3}}. \quad (67)$$

This is just what one expects for gravity coupled to a massless scalar field which behaves just like stiff matter. From the point of view of string theory, we see that we have a time dependent string coupling constant g which increases with time from a zero value at the Big Bang. In other words

$$\boxed{\text{late times} \longleftrightarrow \text{strong coupling}} \quad (68)$$

$$\boxed{\text{early times} \longleftrightarrow \text{weak coupling}} \quad (69)$$

This feature remains true if D0-branes are present. First note that one may take $a(t)$ to be constant by making the change

$$t \rightarrow t + t_0, \quad (70)$$

and letting $t_0 \rightarrow \infty$. One then obtains the static multi-brane solutions

$$ds_{10}^2 = -H^{-\frac{7}{8}} dt^2 + H^{\frac{1}{8}} d\mathbf{x}^2, \quad (71)$$

with

$$H = 1 + \sum_{a=1}^k \frac{\mu_a}{7|\mathbf{x} - \mathbf{x}_a|}, \quad (72)$$

$$g = e^\phi = H^{\frac{3}{4}}, \quad (73)$$

and

$$A = \left(1 - \frac{1}{H}\right) dt. \quad (74)$$

Now let us restore the time dependence. One finds, setting $t_0 = 1$,

$$ds_{10}^2 = -H^{-\frac{7}{8}} dt^2 + t^{\frac{2}{9}} H^{\frac{1}{8}} d\mathbf{x}^2, \quad (75)$$

with

$$H = 1 + \sum_{a=1}^k \frac{\mu_a}{7t^{\frac{16}{9}} |\mathbf{x} - \mathbf{x}_a|}, \quad (76)$$

$$g = e^\phi = t^{\frac{4}{3}} H^{\frac{3}{4}}, \quad (77)$$

and

$$A = \left(1 - \frac{1}{H}\right) \frac{dt}{t}. \quad (78)$$

Evidently the general time-dependent solution represents a gas of D0-branes in 10 dimensions in a background time dependent dilaton field. Note that

- the non-interacting gas of D0-branes does not affect the law of expansion.
- while the solution at large distances is time-dependent, with the physical separation of the D0-branes increasing with time, near each singularity, i.e. as $\mathbf{x} \rightarrow \mathbf{x}_a$, the solution is effectively *static*.

7 Lift to 11 dimensions

The Maki-Shiraishi metrics are clearly not supersymmetric, i.e. they are not BPS because they are time dependent. Every Lorentzian spacetime admitting a Killing spinor field must admit an everywhere non-spacelike Killing vector field. However if one lifts the solution to eleven dimensions using the uplifting formula

$$ds_{11}^2 = e^{-\frac{\phi}{6}} ds_{10}^2 + e^{\frac{4\phi}{3}} (dz + 2A)^2, \quad (79)$$

where z is the eleventh coordinate, something interesting happens.

7.1 The background

Using the uplifting formula one finds that the background (66, 67) becomes

$$ds_{11}^2 = t^{-\frac{2}{9}} (-dt^2 + t^{\frac{2}{9}} d\mathbf{x}^2) + t^{\frac{16}{9}} dz^2. \quad (80)$$

If one defines

$$T = \frac{9}{8} t^{\frac{8}{9}}, \quad (81)$$

we get

$$ds_{11} = -dT^2 + \left(\frac{64}{81}\right)^2 T^2 dz^2 + d\mathbf{x}^2. \quad (82)$$

This is flat space $\mathbb{E}^9 \times \mathbb{E}^{1,1}$ in *Milne coordinates*. Of course if the tenth coordinate z is taken to be periodic, as it would be on the M-Theory circle, then we shall get the usual orbifold singularities and non-Hausdorff behaviour associated with Misner spacetime [18].

7.2 Lifting the general solution

We define

$$d\tilde{t} = t^{\frac{7}{9}} dt \quad \Rightarrow \quad \tilde{t} = \frac{9}{16} t^{\frac{16}{9}}. \quad (83)$$

We obtain

$$ds_{11}^2 = d\mathbf{x}^2 + t^{\frac{16}{9}} H (dz + 2A)^2 - \frac{dt^2}{t^{\frac{2}{9}} H}. \quad (84)$$

Now let

$$d\tilde{z} = dz + \frac{dt}{t}. \quad (85)$$

One gets

$$ds_{11}^2 = d\mathbf{x}^2 + t^{\frac{16}{9}} H (d\tilde{z})^2 - 2d\tilde{z}t^{\frac{7}{9}}. \quad (86)$$

This looks complicated, but if we define a time independent harmonic function

$$\hat{H} = \frac{1}{7} \sum \frac{\mu_a}{|\mathbf{x} - \mathbf{x}_a|^7} \quad (87)$$

and set

$$T = \frac{8}{9} a^8, \quad a = \left(\frac{t + t_0}{t_0} \right)^{\frac{1}{9}}, \quad (88)$$

$$x^0 = T \cosh \left(\frac{8z}{9t_0} \right), \quad x^{10} = T \sinh \left(\frac{8z}{9t_0} \right), \quad (89)$$

$$x^{\pm} = x^0 \pm x^{10} \quad (90)$$

with

$$dz + \frac{dt}{a^9} = t_0 \frac{8}{9} \frac{dx^+}{x^+} \quad (91)$$

we have

$$ds_{11} = d\mathbf{x}^2 - dx^+ dx^- + \hat{H} \left(\frac{dx^+}{x^+} \right)^2. \quad (92)$$

Note that

- This is a pp-wave whose profile depends on light-cone time x^+ .

- The solution is nevertheless *boost-invariant*, the scalings

$$x^+ \rightarrow \lambda x^+, \quad x^- \rightarrow \lambda^{-1} x^- \quad (93)$$

with $\lambda \in \mathbb{R} \setminus 0$, leave the metric invariant.

- Reduction on the boost Killing vector gives the 10-dimensional solution.
- The 11-dimensional solution is BPS, it admits a covariantly constant Killing spinor but this is *not* invariant under boost and hence the 10-dimensional solution is not BPS.

8 Conclusions

- Homothetic solutions of classical matrix theory resemble expanding universes but do not really capture the cosmology of D0-branes.
- Exact supergravity 10-dimensional Type IIA solutions for expanding universes of D0-particles are available.
- Lifted to 11-dimensions they are vacuum pp-wave solutions with time dependent profile and hence BPS. Their reduction to 10-dimensions is on a boost Killing field and hence they are time dependent and non-BPS in 10 dimensions.
- It seems that Quantum Mechanical matrix theory in a suitable limit captures the behaviour of the classical super-gravity solutions.
- The status of the ‘Wave function of the Universe’ remains unclear.

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